

Ferromagnetism and Superconductivity in Quark Matter

— Color magnetic superconductivity —

Toshitaka TATSUMI^{1, *)} Tomoyuki MARUYAMA^{2, **)} and Eiji NAKANO^{3, ***)}

¹*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*

²*College of Bioresource Sciences, Nihon University, Fujisawa, 252-8510, Japan*

³*Department of Physics, Tokyo Metropolitan University, 1-1 Minami-Ohsawa,
Hachioji, Tokyo 192-0397, Japan*

(Received)

A coexistent phase of spin polarization and color superconductivity in high-density QCD is studied at zero temperature. The axial-vector self-energy stemming from the Fock exchange term of the one-gluon-exchange interaction has a central role to cause spin polarization. As a significant feature, the Fermi surface is deformed by the axial-vector self-energy and then rotation symmetry is spontaneously broken down. The gap function results in being anisotropic in the momentum space in accordance with the deformation. It is found that spin polarization barely conflicts with color superconductivity, and almost coexists with it.

§1. Introduction

Recently much interest is given for high-density QCD, especially for quark Cooper-pair condensation phenomena at high-density quark matter (called as color superconductivity (CSC)), in connection with, e.g., physics of heavy ion collisions and neutron stars.¹⁾ Its mechanism is similar to the BCS theory for the electron-phonon system and the quark-quark interaction is mediated by colored gluons in CSC; the color anti-symmetric $\bar{3}$ channel is the most attractive one.

In this talk we would like to address another aspect expected in quark matter: spin polarization or ferromagnetism (FM) of quark matter. A possibility of ferromagnetism in quark matter has been first suggested by one of the authors (T.T.) by the use of the OGE interaction;²⁾ a trade-off between the kinetic and the Fock exchange energies gives rise to spin polarization, similar to Bloch's idea for itinerant electrons.^{3),4)} Salient features of spin polarization in the relativistic system are also discussed in Ref. 2).

If these phases are realized in quark matter, there should be some interplay between them; we examine here a possibility of the coexistence of FM and CSC in quark matter. It would be worth mentioning in this context that ferromagnetism (or spin polarization) and superconductivity are fundamental concepts in condensed matter physics, and their coexistent phase has been discussed for a long time.⁵⁾ As a recent progress, a superconducting phase have been discovered in ferromagnetic materials

*) E-mail: tatsumi@ruby.scphys.kyoto-u.ac.jp

**) E-mail: tomo@brs.nihon-u.ac.jp

***) E-mail: enakano@comp.metro-u.ac.jp

and many efforts have been made to understand the coexisting mechanism.⁶⁾

In a phenomenological context their coexistence should have some implications for the magnetic and thermal properties of compact stars. Recently, a new type of neutron stars, called “magnetars”, with a super strong magnetic field of $\sim O(10^{15}\text{G})$ has been discovered.⁷⁾ They may remind us of a long-standing problem about the origin of the magnetic field in compact stars, since its strength is too large to regard it as a successor from progenitor main-sequence stars, unlike canonical neutron stars.⁸⁾ Since hadronic matter develops over inside neutron stars beyond the nuclear density ($\rho_0 \sim 0.16\text{fm}^{-3}$), it should be interesting to consider the microscopic origin of the magnetic field in magnetars.

Thus, it might be also interesting to examine the possibility of the spin-polarized phase with CSC in quark matter, in connection with magnetars.

We investigate spin polarization in the color superconducting phase by a self-consistent framework, in which quark Cooper pairs are formed under the axial-vector mean-field. We shall see that this phenomenon is a manifestation of spontaneous breaking of both color $SU(3)$ and rotation symmetries.

§2. Ferromagnetism in quark matter

A possibility of FM in quark matter has been first suggested by a perturbative calculation with the one-gluon-exchange (OGE) interaction:²⁾ the mechanism of *spontaneous spin polarization* of quark matter is caused by the Fock exchange interaction, since quark matter is globally color neutral system, which is analogous to that due to itinerant electrons in condensed matter physics.^{3),4)} One of the important features there is that there is no spin independent interaction *ab initio*, while the spin polarization occurs as a consequence of the Pauli exclusion principle: electrons with the same spin direction can effectively avoid the Coulomb repulsion. Since a quark should be treated relativistically and no more eigenstate of the spin operator Σ_z , we must carefully define “ferromagnetism” in quark matter.²⁾

By operating the Fierz transformation on the original Lagrangian, we can extract the relevant Fock interaction to magnetism: we shall see the condensation of the axial-vector mean-field (AV), $\langle \bar{\psi} \gamma_5 \gamma_3 \psi \rangle$, is a key phenomenon.⁹⁾ It is well known that any non-vanishing mean-field, $\langle \bar{\psi} \Gamma_\alpha \psi \rangle$, implies the importance of the particle-hole (or anti-particle) degree of freedom.

Taking AV in the form,

$$\mathbf{V} = -\gamma_5 \gamma_3 \mathbf{U}_A, \quad \mathbf{U}_A // \hat{z}, \quad (2.1)$$

without loss of generality, we have the quark Green function in the presence of AV,

$$G_A^{-1}(p) = \not{p} - m - \not{\mu} + \gamma_5 \not{U}_A. \quad (2.2)$$

The particle spectrum is then easily obtained by seeking for the poles of the Green function $G_A(p)$, $\det G_A^{-1}(p_0 = \epsilon_n)|_{\mu=0} = 0$, for uniform magnetization:

$$\epsilon_n = \pm \epsilon_{\pm} \quad (2.3)$$

$$\epsilon_{\pm} = \sqrt{\mathbf{p}^2 + \mathbf{U}_A^2 + m^2 \pm 2\sqrt{m^2 \mathbf{U}_A^2 + (\mathbf{p} \cdot \mathbf{U}_A)^2}}, \quad (2.4)$$

where the sign \pm in ϵ_{\pm} represents an *exchange splitting* of different “spin” eigenstates.⁴⁾ The spectrum is reduced to a familiar form $\epsilon_{\pm} \sim m + \frac{p^2}{2m} \pm |\mathbf{U}_A|$ in the non-relativistic limit.⁴⁾ In the massless limit, on the other hand, $\epsilon_{\pm} \rightarrow [p_{\perp}^2 + (|p_z| \pm |\mathbf{U}_A|)^2]^{1/2}$, which is nothing else but the free-particle energy with a shift of the z component of momentum, and U_A becomes a redundant degree of freedom. We can immediately see that there appears a coupling term in ϵ_{\pm} , $\propto \mathbf{p} \cdot \mathbf{U}_A$, reflecting that the direction of spin changes as a particle moves in the relativistic case. Thus we must consider the spin configuration in the phase space, different from the non-relativistic case like the Heisenberg model.

The system loses rotation symmetry in the momentum space as well as in the coordinate space due to the presence of AV ($O(3) \rightarrow O(2)$), so that the Fermi sea is deformed in the relativistic case. Consequently we have two deformed Fermi seas with different volumes: one (F^-) has a prolate shape for majority particles and the other (F^+) an oblate shape for minority particles. *)

The mean-value of the spin operator is then given by

$$\begin{aligned} \bar{s}_z &= \frac{1}{2} \langle \Sigma_z \rangle = -i \int_C \frac{d^4 p}{(2\pi)^4} \text{tr} \gamma_5 \gamma_3 G_A(p) \\ &= \frac{1}{2} \left[\int_{F^+} \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{U}_A| + \beta}{\epsilon_+} + \int_{F^-} \frac{d^3 p}{(2\pi)^3} \frac{|\mathbf{U}_A| - \beta}{\epsilon_-} \right] \end{aligned} \quad (2.5)$$

with $\beta = \sqrt{p_z^2 + m^2}$, and we can obviously see that the nonvanishing value of $|\mathbf{U}_A|$ causes finite magnetization.

§3. Color magnetic superconductivity

Here let's tackle the coexistence problem of FM and CSC, a possibility of *color magnetic superconductivity*;¹¹⁾ more definitely, our main concern here is to examine the possibility of the quark Cooper instability under AV in Eq. (2.1), which is responsible for spin polarization of quark matter. In other words we'd like to figure

*) It would be interesting to see a similar idea given in ref. 10); these authors *variationally* introduced the deformed Fermi seas in the context of the gap function of CSC for a pair with different Fermi surfaces. Their shapes look very similar to our case, but its origin is completely different to each other.

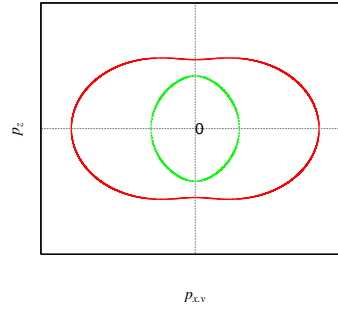


Fig. 1. Schematic view of the shapes of the Fermi seas: a prolate shape for majority “spin” quarks (F^-) and an oblate shape for minority “spin” quarks (F^+).

out the interplay between particle-particle and particle-hole degrees of freedom; the former leads to CSC, while the latter FM. We start with the OGE action in QCD:

$$I_{int} = -g^2 \frac{1}{2} \int d^4x \int d^4y \left[\bar{\psi}(x) \gamma^\mu \frac{\lambda_a}{2} \psi(x) \right] D_{\mu\nu}(x, y) \left[\bar{\psi}(y) \gamma^\nu \frac{\lambda_a}{2} \psi(y) \right], \quad (3.1)$$

Using the mean-field approximation, we have

$$I_{MF} = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \begin{pmatrix} \bar{\psi}(p) \\ \bar{\psi}_c(p) \end{pmatrix}^T G^{-1}(p) \begin{pmatrix} \psi(p) \\ \psi_c(p) \end{pmatrix}, \quad (3.2)$$

in the Nambu-Gorkov formalism. Assuming the color singlet particle-hole pair mean-field, V , as well as the color $\bar{3}$ particle-particle pair mean-field, Δ , the inverse Green function G^{-1} renders

$$\begin{aligned} G^{-1}(p) &= \begin{pmatrix} \not{p} - m + \not{\mu} + V(p) & \gamma_0 \Delta^\dagger(p) \gamma_0 \\ \Delta(p) & \not{p} - m - \not{\mu} + \bar{V}(p) \end{pmatrix}, \\ &= \begin{pmatrix} G_{11}(p) & G_{12}(p) \\ G_{21}(p) & G_{22}(p) \end{pmatrix}^{-1} \end{aligned} \quad (3.3)$$

where charge-conjugated spinor ψ_c and mean-field \bar{V} are defined by

$$\psi_c(k) = C \bar{\psi}^T(-k), \quad \bar{V} \equiv C V^T C^{-1}. \quad (3.4)$$

Then we immediately have the coupled equations, representing the interplay of two different mean-fields, V and Δ :

$$-V(k) = (-ig)^2 \int \frac{d^4p}{i(2\pi)^4} \{ -i D^{\mu\nu}(k-p) \} \gamma_\mu \frac{\lambda_\alpha}{2} \{ -i G_{11}(p) \} \gamma_\nu \frac{\lambda_\alpha}{2}. \quad (3.5)$$

$$-\Delta(k) = (-ig)^2 \int \frac{d^4p}{i(2\pi)^4} \{ -i D^{\mu\nu}(k-p) \} \gamma_\mu \frac{-(\lambda_\alpha)^T}{2} \{ -i G_{21}(p) \} \gamma_\nu \frac{\lambda_\alpha}{2}. \quad (3.6)$$

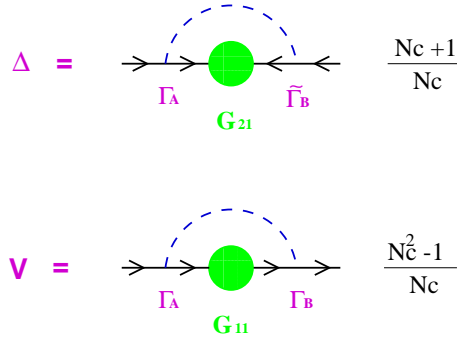


Fig. 2. Graphical interpretations of the coupled equations (3.5) and (3.6) with coefficients in front of R.H.S. given by N_c .

Applying the Fierz transformation for the self-energy term (3.5) we can see that there appear the color-singlet scalar, pseudoscalar, vector and axial-vector self-energies. In general we must take into account these self-energies in V , $V = U_s + \gamma_5 U_{ps} + \gamma_\mu U_v^\mu + \gamma_\mu \gamma_5 U_{av}^\mu$ with the mean-fields U_i . Here we retain only U_s, U_v^0, U_{av}^3 in V and suppose that others to be vanished.

We shall see this ansatz gives self-consistent solutions for Eq.(3.5) because of axial and reflection symmetries of the Fermi seas under the zero-range approximation for the gluon propagator (see §3.1). We furthermore discard the scalar mean-field U_s and the time component of the vector mean-field U_v^0 for simplicity since they are directly irrelevant for the spin degree of freedom.

Here we take the following ansatz for V and Δ : V has a form given in Eq. (2.1) and

$$\Delta(\mathbf{p}) = \sum_s \tilde{\Delta}_s(\mathbf{p}) B_s(\mathbf{p}), \quad B_s(\mathbf{p}) = \gamma_0 \phi_{-s}(\mathbf{p}) \phi_s^\dagger(\mathbf{p}) \quad (3.7)$$

with the energy eigenspinors ϕ_s , $s = \pm$ with eigen energies (2.4), ϵ_s , s.t. $G_A^{-1}(\epsilon_s, \mathbf{p}) \phi_s = 0$. The structure of the gap function (3.7) is inspired by a physical consideration of a quark pair on the same Fermi surface of F^- or F^+ :¹¹⁾ we consider here the quark pair on each Fermi surface with opposite momenta, \mathbf{p} and $-\mathbf{p}$ so that they result in a linear combination of $J^\pi = 0^-, 1^-$ (see Fig. 3). *) $\tilde{\Delta}_s$ is still a matrix in the color-flavor space and we take

$$\left(\tilde{\Delta}_s \right)_{\alpha\beta;ij} = \epsilon^{\alpha\beta 3} \epsilon^{ij} \Delta_s \quad (3.8)$$

to satisfy the fermionic constraint for the two-flavor case (2SC), where α, β denote the color indices and i, j the flavor indices. Then the quasi-particle spectrum can be obtained by looking for poles of the diagonal Green function, G_{11} :

$$E_s(\mathbf{p}) = \begin{cases} \sqrt{(\epsilon_s(\mathbf{p}) - \mu)^2 + |\Delta_s(\mathbf{p})|^2} & \text{for color 1, 2} \\ \sqrt{(\epsilon_s(\mathbf{p}) - \mu)^2} & \text{for color 3} \end{cases} \quad (3.9)$$

Note that the quasi-particle energy is independent of color and flavor in this case, since we have assumed a singlet pair in flavor and color. Gathering all these stuffs to put them in the self-consistent equations (3.5), (3.6), we definitely write down the coupled equations for Δ_s ,

$$\Delta_{s'}(k, \theta_k) = \frac{N_c + 1}{2N_c} \tilde{g}^2 \int \frac{dp d\theta_p}{(2\pi)^2} p^2 \sin \theta_p \sum_s T_{s's}(k, \theta_k, p, \theta_p) \frac{\Delta_s(p, \theta_p)}{2E_s(p, \theta_p)}, \quad (3.10)$$

and for U_A ,

$$U_A = -\frac{N_c^2 - 1}{4N_c^2} \tilde{g}^2 \int \frac{d^3 p}{(2\pi)^3} \sum_s \{ \theta(\mu - \epsilon_s(\mathbf{p})) + 2v_s^2(\mathbf{p}) \} \frac{U_A + s\beta_p}{\epsilon_s(\mathbf{p})}, \quad (3.11)$$

*) Note that this choice is not unique; actually we are now studying another possibility of quark pair between different Fermi surfaces.¹²⁾

within the “contact” interaction with $\tilde{g}^2 = g^2/\Lambda^2$ (see Eq. (3.13)), where $v_s^2(\mathbf{p})$ denotes the momentum distribution of the quasi-particles (cf. Eq. (2.5)). Carefully analyzing the structure of the function $T_{s's}$ in Eq. (3.10), we can easily find that the gap function s.t. Δ_s should have the polar angle (θ) dependence on the Fermi surface s.t.

$$\Delta_s(p_s^F, \theta) = \frac{p_s^F(\theta) \sin \theta}{\mu} \left(-s \frac{m}{\sqrt{m^2 + (p_s^F(\theta) \cos \theta)^2}} R + F \right), \quad (3.12)$$

with constants F and R to be determined. As a characteristic feature, both the gap

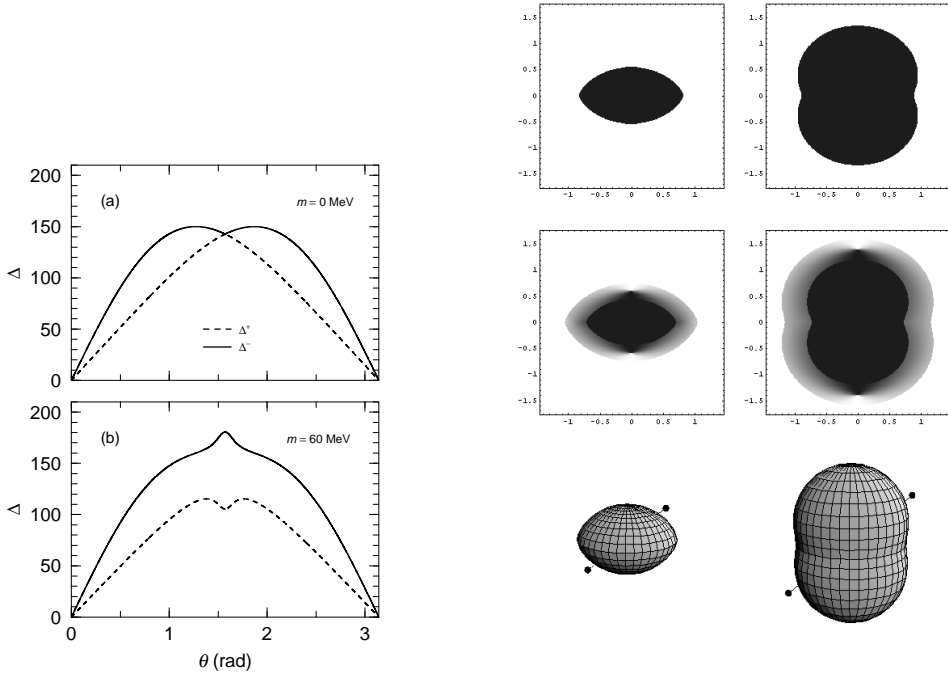


Fig. 3. (left panel) Schematic view of the polar-angle dependence of the gap functions at the Fermi surface: (a) for $m = 0$ and (b) for $m \neq 0$. (right panel) Deformed Fermi seas and quark pair on each surface. The middle two figures show diffusion of the Fermi surfaces in the presence of the polar-angle dependent gap function Δ_{\pm} .

functions have nodes at poles ($\theta = 0, \pi$) and take the maximal values at the vicinity of equator ($\theta = \pi/2$), keeping the relation, $\Delta_- \gtrsim \Delta_+$ (Fig. 3). This feature is very similar to the 3P pairing in liquid ^3He or nuclear matter;^{13), 14)} actually we can see our gap function Eq. (3.12) to exhibit an effective P wave nature by a genuine relativistic effect due to the Dirac spinors. To summarize, we depict in Fig. (3) the deformed Fermi seas for $\Delta_{\pm} = 0$ and the quasi-particle distributions for $\Delta_{\pm} \neq 0$.

We also find that the expression for U_A , Eq. (3.11), is nothing but the simple sum of the expectation value of the spin operator with the weight of the occupation probability of the quasi-particles v_s^2 for two colors and the step function for remaining one color (cf. (2.5)).

3.1. Self-consistent solutions

Here we demonstrate some numerical results; we replaced the original OGE by the “contact” interaction with the cutoff around the Fermi surface in the momentum space,

$$D^{\mu\nu} \rightarrow -g^{\mu\nu}/\Lambda^2, \quad \Delta_s(\mathbf{p}) \rightarrow \Delta_s(\mathbf{p})\theta(\delta - |\epsilon_s - \mu|) \quad (3.13)$$

as in the BCS theory in the weak-coupling limit.

First we show the magnitude of AV. It is seen that the axial-vector mean-field (spin polarization) appears above a critical density and becomes larger as baryon number density gets higher. Moreover, the results for different values of the quark mass show that spin polarization grows more for the larger quark mass. This is because a large quark mass gives rise to much difference in the Fermi seas of two different “spin” states, which leads to growth of the exchange energy in the axial-vector channel. A slight reduction of U_A arises as a result of diffuseness of the Fermi surface due to Δ_s . As seen in Eq. (3.11), U_A can be obtained as addition and cancellation of the contributions by two different Fermi seas; the latter term is more momentum dependent than the former one and thereby $v_s^2(\mathbf{p})$ enhances the cancellation term (see Fig. 4).

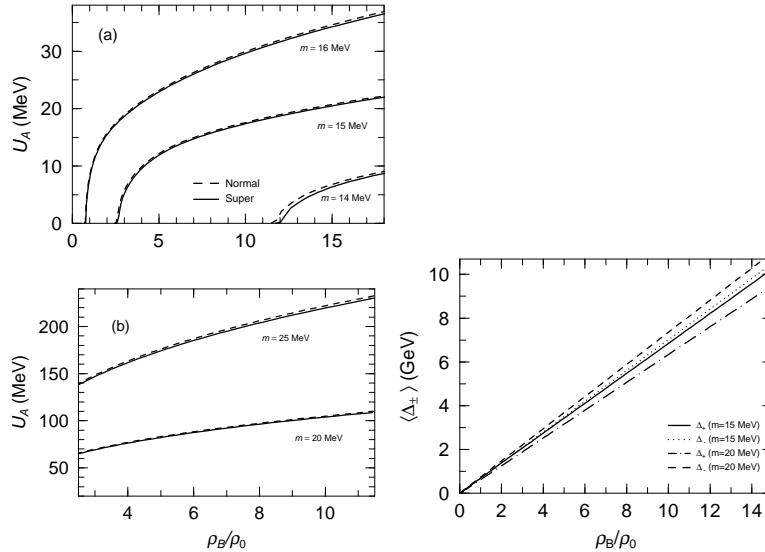


Fig. 4. (Left panel) Axial-vector mean-field (AV) as a function of baryon number density ρ_B ($\rho_0 = 0.16\text{fm}^{-3}$). Solid (dashed) lines denote AV in the presence (absence) of CSC; (Right panel) Mean value of the gap functions.

Next we show the gap function as a function of ρ_B . To see the bulk behavior of the gap function, we use the mean-value with respect to the polar angle on the Fermi surface, $\langle \Delta_{\pm} \rangle \equiv (\int_0^\pi d\theta \sin\theta \Delta_{\pm}^2/2)^{1/2}$. The mean values $\langle \Delta_{\pm} \rangle$ begin to split at a certain density where U_A becomes finite.

With these figures we can say that FM and CSC barely interfere with each other.

§4. Summary and concluding remarks

We have discussed color magnetic superconductivity with a simple model and found that the coexistence of FM and CSC is possible, while CSC somewhat suppresses FM.

We can still expect a high magnetic field enough to explain $B_{\text{MAX}} \sim 10^{15} \text{G}$ observed in magnetars, while the magnitude of polarization is not so large (\sim several %).

Through numerical calculations we used the constant quark masses by regarding them as input parameters and studied the mass effect on the present problem. We found that the quark mass is very important to realize, especially, FM, and a larger quark mass favors spontaneous magnetization. Once taking into account the restoration of chiral symmetry at a certain density, we must allow a dynamical change of the quark mass. This subject is under progress.¹⁵⁾

Finally it would be worth mentioning related other works. As a recent progress in CSC, some authors considered $S = 1$ pairing as well as a usual one,¹⁶⁾ which also exhibits an polar-angle dependence of the gap function. However, its magnitude is much less than the usual one. Note that there is only one type of pairing in our case.

We hope this work serves as an impetus for further study on color magnetic superconductivity.

Acknowledgements

The present work of T.T. and T.M. is partially supported by the Japanese Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology (11640272, 13640282), and by the REIMEI Research Resources of Japan Atomic Energy Research Institute (JAERI).

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